Pascal's Theorem

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Abstract

Pascal's Theorem, a.k.a. *hexagrammum mysticum theorem*, is an interesting theorem about conic sections. The Appendix A of [1] has provided a formal proof of Pascal's Theorem. In this article, we discuss the proof informally and make the material more accessible.

1 Projective Geometry

The world is better if there is exactly one intersection point for any two distinct straight lines. Unfortunately, it is not true for parallel lines in Euclidean geometry. We may change the world by introducing *projective geometry*. In this new world, two parallel lines intersect at a point at infinity.

We represent a 2-dimensional Euclidean point (x, y) by a triple

$$(X, Y, Z), \qquad \text{where } Z \neq 0, \quad x = X/Z \quad \text{and} \quad y = Y/Z.$$

There are more than one equivalent triples representing the same point (x, y). Two triples (X, Y, Z) and (A, B, C) are equivalent if and only if

$$X = kA, \quad Y = kB \quad \text{and} \quad Z = kC \qquad \text{for some } k \neq 0.$$

1.1 Points at Infinity

When $Z = 0$ and either $X \neq 0$ or $Y \neq 0$, the point $(X, Y, 0)$ is *a point at infinity*. Note the triple $(0, 0, 0)$ does not represent any points. Vertical lines pass through the point at infinity $(0, 1, 0)$, or equivalently, $(0, Y, 0)$ for any $Y \neq 0$. Any lines with a finite slope m pass through the point at infinity $(1, m, 0)$, or equivalently, $(X, mX, 0)$ for any $X \neq 0$. Note that there are infinitely many points at infinity. The points at infinity form a straight line, called *the line at infinity*.

1.2 Intersections of Projective Curves

After added the points at infinity, any two distinct straight lines intersect at exactly one point with no exceptions. It can be generalized to the higher degree curves as in Theorem 1 below.

Definition (Degree of a Curve). Let $C : f(x, y) = 0$ be a curve, where $f(x, y)$ is a polynomial in x and y with degree[1] d. The degree of C, denoted by $\deg C$, is also d.

The projective version (homogenization) of C is

$$F(X, Y, Z) \quad := \quad Z^d f(X/Z, Y/Z) \quad = \quad 0.$$

$F(X, Y, Z)$ is called a *homogeneous polynomial*, where all the terms have the same degree d.

Example. Let $L_1 : 3x + y + 1 = 0$ and $L_2 : 3x + y + 2 = 0$. The homogeneous equations are

$$L_1 : 3X + Y + Z = 0 \qquad \text{and} \qquad L_2 : 3X + Y + 2Z = 0.$$

These parallel lines intersect at the point at infinity $(1, -3, 0)$, which is a solution to both equations.

[1] It counts all the variables, e.g. the degree of $x^2y^2 + x^3 + 1$ is 4 and the degree of $X^2Y^2 + X^3Z + Z^4$ is also 4, .

Theorem 1 (Simplified Bezout's Theorem)**.** *For any two projective curves C_1 and C_2 with no common components, the number of complex intersection points (counting multiplicity[2]) is equal to*

$$(\deg C_1)(\deg C_2).$$

As a consequence, the number of real intersection points is less than or equal to $(\deg C_1)(\deg C_2)$.

Proof. See Theorem A.1 (p285) and the proof of it in Section A.4 (p290) [1]. □

Theorem 2 (Simplified Cayley-Bacharach Theorem)**.** *Let C_1 and C_2 be projective curves with degree 3. Suppose they have 9 distinct intersection points. Let D be another cubic curve. If D passes through any 8 out of 9 intersection points of C_1 and C_2, then D must pass through the ninth intersection point.*

Proof. See the proof of Theorem A.3 (p288) in [1]. See also Theorem A.2 (p288) for a generalization. □

2 Pascal's Theorem

Once we have equipped the required theorems, proving Pascal's theorem is straightforward. Note that we assume (real or complex) projective geometry in this section. Any points are possibly a point at infinity. Pascal's theorem also applies to Euclidean geometry after excluding the parallel line cases.

Theorem 3 (p289, Pascal's Theorem)**.** *Let C be a degree 2 projective curve, i.e. a conic section. Let P_1, P_2, P_3, P_4, P_5 and P_6 be 6 distinct points on C in any order. Suppose*

(i) *the lines $\overline{P_1P_2}$ and $\overline{P_4P_5}$ intersect at point Q_1,*

(ii) *the lines $\overline{P_2P_3}$ and $\overline{P_5P_6}$ intersect at point Q_2, and*

(iii) *the lines $\overline{P_3P_4}$ and $\overline{P_6P_1}$ intersect at point Q_3.*

Then, Q_1, Q_2 and Q_3 are collinear.

Proof. The lines $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, $\overline{P_4P_5}$, $\overline{P_5P_6}$ and $\overline{P_6P_1}$ are distinct by Theorem 1. Otherwise, if any two of the lines are the same, the conic C passes through more than 2 points on that line. Similarly, the points Q_1, Q_2 and Q_3 are distinct and not on C by Theorem 1. WLOG, we show the cases below.

(a) Distinct: Suppose $Q_1 = Q_2$. Then $\overline{P_1P_2}$ and $\overline{P_2P_3}$ have 2 intersection points P_2 and Q_1.

(b) Not on C: Suppose Q_3 is on C. Then $\overline{P_3P_4}$ and C have 3 intersection points P_3, P_4 and Q_3.

Construct[3] two degree 3 curves:

$$C_1: \ \overline{P_1P_2} \cup \overline{P_3P_4} \cup \overline{P_5P_6} \qquad \text{and} \qquad C_2: \ \overline{P_2P_3} \cup \overline{P_4P_5} \cup \overline{P_6P_1}.$$

By Theorem 1, C_1 and C_2 have at most 9 intersection points. By (a) and (b), they have exactly 9 distinct intersection points P_1, P_2, P_3, P_4, P_5, P_6, Q_1, Q_2 and Q_3. Construct another degree 3 curve

$$D: \quad C \cup \overline{Q_1Q_2}.$$

Then, D passes through 8 points P_1, P_2, P_3, P_4, P_5, P_6, Q_1 and Q_2. By Theorem 2, D passes through Q_3. Finally, Q_3 is on $\overline{Q_1Q_2}$ since Q_3 is not on C by (b). □

References

[1] Joseph H. Silverman and John T. Tate. *Rational Points on Elliptic Curves.* Springer Publishing Company, Incorporated, 2nd edition, 2015.

[2] A rigorous definition of *intersection multiplicity* is non-trivial; see p294 - p295.

[3] Given two curves

$$C_a : F(X,Y,Z) = 0 \qquad \text{and} \qquad C_b : G(X,Y,Z) = 0.$$

We may construct the union curve as

$$C_a \cup C_b : \quad F(X,Y,Z)\,G(X,Y,Z) = 0.$$