Construction of Pentagon & Heptadecagon

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1 Pentagon

Let $\zeta_5 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ be a 5th root of unity. Then,

$$\zeta_5^5 = 1 \qquad \text{and} \qquad \zeta_5^2 + \zeta_5 + 1 + \zeta_5^{-1} + \zeta_5^{-2} = 0.$$

Let $\alpha_5 = \zeta_5 + \zeta_5^{-1}$. Then, $\alpha_5^2 = \zeta_5^2 + 2 + \zeta_5^{-2}$. We have

$$\alpha_5^2 + \alpha_5 - 1 = 0.$$

Since $\alpha_5 > 0$,

$$\alpha_5 = \frac{-1+\sqrt{5}}{2} \approx 0.618034.$$

We also have

$$\zeta_5^2 - \alpha_5\zeta_5 + 1 = 0.$$

Since $\Im(\zeta_5) > 0$,

$$\zeta_5 = \frac{\alpha_5 + \sqrt{\alpha_5^2 - 4}}{2} = \frac{\alpha_5 + i\sqrt{\alpha_5 + 3}}{2}.$$

Finally,

$$\begin{aligned} \cos\frac{2\pi}{5} &= \Re(\zeta_5) &&= \frac{\alpha_5}{2} &&= \frac{-1+\sqrt{5}}{4} &&\approx 0.309017; \\ \sin\frac{2\pi}{5} &= \Im(\zeta_5) &&= \frac{\sqrt{\alpha_5+3}}{2} &&= \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}} &&\approx 0.951057. \end{aligned}$$

2 Heptadecagon

Similarly, let $\zeta = \zeta_{17} = \cos\frac{2\pi}{17} + i\sin\frac{2\pi}{17}$ be a 17th root of unity. We have,

$$\zeta^{17} = 1 \qquad \text{and} \qquad \sum_{k=-8}^{8} \zeta^k = 0.$$

Let

$$\begin{aligned} \alpha &= \zeta + \zeta^{-1}, \\ \beta &= \zeta^4 + \zeta + \zeta^{-1} + \zeta^{-4}, \\ \gamma &= \zeta^8 + \zeta^4 + \zeta^2 + \zeta + \zeta^{-1} + \zeta^{-2} + \zeta^{-4} + \zeta^{-8}. \end{aligned}$$

Compute γ^2 in terms of ζ. We have the following table.

	ζ^8	ζ^7	ζ^6	ζ^5	ζ^4	ζ^3	ζ^2	ζ	1
γ	1	0	0	0	1	0	1	1	0
γ^2	3	4	4	4	3	4	3	3	8

Note that the coefficient of ζ^{-k} is equal to the coefficient of ζ^k for $1 \le k \le 8$. We have,

$$\gamma^2 + \gamma - 4 = 0.$$

Since $\gamma > 0$,

$$\gamma = \frac{-1+\sqrt{17}}{2} \approx 1.561553.$$

Consider

$$\beta(\gamma - \beta) = (\zeta^4 + \zeta + \zeta^{-1} + \zeta^{-4})(\zeta^8 + \zeta^2 + \zeta^{-2} + \zeta^{-8}) = -1.$$

The equation $x^2 - \gamma x - 1 = 0$ has roots β and $\gamma - \beta$. Since $\beta > 0$,

$$\beta = \frac{\gamma + \sqrt{\gamma^2+4}}{2} = \frac{\gamma + \sqrt{-\gamma+8}}{2} = \frac{1}{2}\left(\frac{-1+\sqrt{17}}{2} + \sqrt{\frac{17-\sqrt{17}}{2}}\right) \approx 2.049481.$$

Compute β^2 and β^3 in terms of γ.

$$\begin{aligned}
\beta^2 &= \frac{1}{2}\left(-\gamma + 6 + \gamma\sqrt{-\gamma+8}\right) \\
\beta^3 &= \frac{1}{4}\left(-\gamma + 6 + \gamma\sqrt{-\gamma+8}\right)\left(\gamma + \sqrt{-\gamma+8}\right) \\
&= \frac{1}{2}\left(8\gamma - 4 + (-\gamma+5)\sqrt{-\gamma+8}\right)
\end{aligned}$$

Compute β^2 β^3 and β^4 in terms of ζ.

	ζ^8	ζ^7	ζ^6	ζ^5	ζ^4	ζ^3	ζ^2	ζ	1
β	0	0	0	0	1	0	0	1	0
β^2	1	0	0	2	0	2	1	0	4
β^3	3	3	3	1	9	1	3	9	0
β^4	16	10	10	24	5	24	16	5	36

We have

$$0 = \beta^4 + \beta^3 - 6\beta^2 - \beta + 1$$

Write $\alpha(\beta - \alpha) = (\zeta + \zeta^{-1})(\zeta^4 + \zeta^{-4}) = \zeta^5 + \zeta^3 + \zeta^{-3} + \zeta^{-5}$ in terms of β using the table above. Then,

$$\alpha(\beta - \alpha) = \frac{1}{26}(3\beta^4 - 10\beta^3 - 18\beta^2 + 75\beta - 36) = \frac{1}{26}(-13\beta^3 + 78\beta - 39) = -\frac{1}{2}(\beta^3 - 6\beta + 3)$$

The equation $0 = x^2 - \beta x - \frac{1}{2}(\beta^3 - 6\beta + 3)$ has roots α and $\beta - \alpha$. Since $\beta > \alpha - \beta$,

$$\begin{aligned}
\alpha &= \frac{1}{2}\left(\beta + \sqrt{\beta^2 + 2(\beta^3 - 6\beta + 3)}\right) \\
&= \frac{1}{2}\left(\beta + \sqrt{8\gamma - 4 + (-\gamma+5)\sqrt{-\gamma+8} + \frac{1}{2}\left(-\gamma + 6 + \gamma\sqrt{-\gamma+8}\right) - 6\left(\gamma + \sqrt{-\gamma+8}\right) + 6}\right) \\
&= \frac{1}{2}\left(\beta + \sqrt{\frac{3\gamma}{2} + 5 - \left(\frac{\gamma}{2} + 1\right)\sqrt{-\gamma+8}}\right) \\
&= \frac{1}{4}\left(\frac{-1+\sqrt{17}}{2} + \sqrt{\frac{17-\sqrt{17}}{2}} + \sqrt{17 + 3\sqrt{17} - \left(3+\sqrt{17}\right)\sqrt{\frac{17-\sqrt{17}}{2}}}\right) \\
&\approx 1.864944.
\end{aligned}$$

Note that

$$\alpha^8 + \alpha^7 - 7\alpha^6 - 6\alpha^5 + 15\alpha^4 + 10\alpha^3 - 10\alpha^2 - 4\alpha + 1 = 0.$$

Since $\zeta^2 - \alpha\zeta + 1 = 0$ and $\Im(\zeta) > 0$,

$$\zeta = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2}.$$

Finally,

$$\begin{aligned}
\cos\frac{2\pi}{17} &= \Re(\zeta) = \frac{\alpha}{2} \approx 0.932472; \\
\sin\frac{2\pi}{17} &= \Im(\zeta) = \frac{1}{2}\sqrt{4 - \alpha^2} \approx 0.361242.
\end{aligned}$$